Review of Matrix Algebra

Matrix Algebra

In this course, we need to solve system of linear equations in the form

$$a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1n}\,x_n = b_1$$

$$a_{12}\,x_1 + a_{22}\,x_2 + \ldots + a_{2n}\,x_n = b_2$$

$$\ldots\ldots\ldots.. \quad \ldots\ldots\ldots\ldots\ldots. \quad \ldots\ldots\ldots.$$

$$a_{n1}\,x_1 + a_{n2}\,x_2 + \ldots + a_{nn}\,x_n = b_n \qquad \text{(Eqn 1)}$$

where x_1, x_2, …, x_n are the unknowns.

Eqn. (1) can be written in a matrix form as

$$[A]\{x\} = \{b\} \qquad \text{(eqn 2)}$$

where [A] is a (n x n) square matrix, {x} and {b} are (n x 1) vectors

The square matrix [A] and the {x} and {b} vectors are given by,

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}, \quad \text{and} \quad \{b\} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

Note:
Element located at ith row and jth column of matrix [A] is denoted by a_{ij}.
For example, element at the 2nd row and 2nd column is a_{22}.

Matrix Multiplication

The product of matrix [A] of size (m x n) and matrix [B] of size (n x p) will results in matrix [C], with size (m x p).

$$\underset{(m \times n)}{[A]} \quad \underset{(n \times p)}{[B]} = \underset{(m \times p)}{[C]}$$

Note: The (ij)th component of [C], i.e. c_{ij}, is obtained by taking the DOT product,

$$c_{ij} = (i\text{th row of } [A])\,(j\text{th column of } [B])$$

Example:

$$\underset{(2 \times 3)}{\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}} \underset{(3 \times 2)}{\begin{bmatrix} 1 & 4 \\ 5 & -2 \\ 0 & 3 \end{bmatrix}} = \underset{(2 \times 2)}{\begin{bmatrix} 7 & 15 \\ -10 & 7 \end{bmatrix}}$$

Matrix Transposition

If matrix [A] = [a_{ij}], then transpose of [A], denoted by $[A]^T$, is given by $[A]^T = [a_{ji}]$. Thus, the rows of [A] becomes the columns of $[A]^T$.

Example:

$$[A] = \begin{bmatrix} 1 & -5 \\ 0 & 6 \\ -2 & 3 \\ 4 & 2 \end{bmatrix}$$

Then,

$$[A]^T = \begin{bmatrix} 1 & 0 & -2 & 4 \\ -5 & 6 & 3 & 2 \end{bmatrix}$$

Note: In general, if [*A*] is of dimension (*m* x *n*), then $[A]^T$ has the dimension of (*n* x *m*)

Transpose of a Product

The transpose of a product of matrices is given by the product of the transposes of each matrices, in reverse order, i.e.

$([A][B][C])^T \; [C]^T [B]^T [A]^T$

Determinant of a Matrix

Consider a 2 x 2 square matrix [*x*],

$$[x] = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

The determinant of this matrix is give by,

$$det\,[x] = x_{11}x_{22} - x_{21}x_{12}$$

Solution of System of Linear Equations

System of linear algebraic equations can be solved for the unknown using the following methods:

a) Cramer's Rule
b) Inversion of Coefficient Matrix
c) Gaussian Elimination**
d) Gauss-Seidel Iteration

Example: Solve the following SLEs using Gaussian elimination.

$$2x_1 + 1x_2 - 3x_3 = 11 \qquad \text{(i)}$$

$$4x_1 - 2x_2 + 3x_3 = 8 \qquad \text{(ii)}$$

$$-2x_1 + 2x_2 - 1x_3 = -6 \qquad \text{(iii)}$$

Solution

Eliminate x_1 from eq.(ii) and eq.(iii). Multiply eq.(ii) by 0.5 we get

$$
\begin{aligned}
2x_1 + 1x_2 - 3x_3 &= 11 && \text{(i)} \\
2x_1 - 1x_2 + 1.5x_3 &= 4 && \text{(ii)}\ * \\
-2x_1 + 2x_2 - 1x_3 &= -6 && \text{(iii)}
\end{aligned}
$$

Subtract eq.(from eq.(i), we obtain

$$
\begin{aligned}
2x_1 + 1x_2 - 3x_3 &= 11 && \text{(i)} \\
0x_1 + 2x_2 - 4.5x_3 &= 7 && \text{(ii)}\ ** \\
-2x_1 + 2x_2 - 1x_3 &= -6 && \text{(iii)}
\end{aligned}
$$

Add eq.(iii) with eq.(i), yields

$$
\begin{aligned}
2x_1 + 1x_2 - 3x_3 &= 11 && \text{(i)} \\
0x_1 + 2x_2 - 4.5x_3 &= 7 && \text{(ii)}\ ** \\
0x_1 + 3x_2 - 4x_3 &= 5 && \text{(iii)}\ *
\end{aligned}
$$

Eliminate x_2 from eq.(iii)*. Multiply eq.(ii)** by 3 and eq.(iii)* by 2 we get

$$
\begin{aligned}
2x_1 + 1x_2 - 3x_3 &= 11 && \text{(i)} \\
0x_1 + 6x_2 - 13.5x_3 &= 21 && \text{(ii)}\ *** \\
0x_1 + 6x_2 - 8x_3 &= 10 && \text{(iii)}\ **
\end{aligned}
$$

Subtract eq.(iii)** from eq.(ii)***, we obtain

$$
\begin{aligned}
2x_1 + 1x_2 - 3x_3 &= 11 && \text{(i)} \\
0x_1 + 2x_2 - 4.5x_3 &= 7 && \text{(ii)}\ ** \\
0x_1 + 0x_2 - 5.5x_3 &= 11 && \text{(iii)}\ ***
\end{aligned}
$$

From eq.(iii)*** we determine the value of x_3, i.e

$$X_3 = \frac{11}{-5.5} = -2$$

Back substitute value of x_3 into eq.(ii)** and solve for x_2, we get

$$X_2 = \frac{7+4.5(-2)}{2} = -1$$

Back substitute value of x_2 and x_3 into eq.(i) and solve for x_1, we get

$$X_1 = 3$$