INTRODUCTION

In order to understand the theory of reinforced concrete beams you would need to study a very wide range of material well beyond the scope of one part of one outcome of one module. For this reason the material offered here is greatly simplified.

ELASTIC BENDING

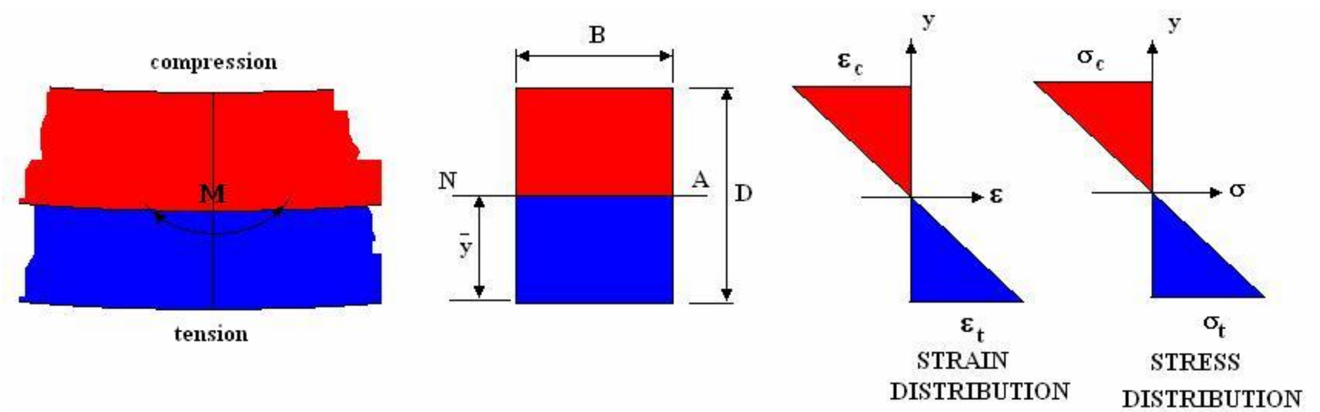
Consider a simple rectangular beam at a point where the bending moment is M.

The neutral axis passes through the centre of area.

The strain is directly proportional to distance y.

Since $\sigma = E\,\varepsilon$ the stress is also directly proportional to distance y.

Suffix 'c' refers to compression and 't' to tension.



From the bending formula we know that the stress at the outer edge is $\sigma = M\,y/I = M/Z$

$y = D/2$ at the outer edges.

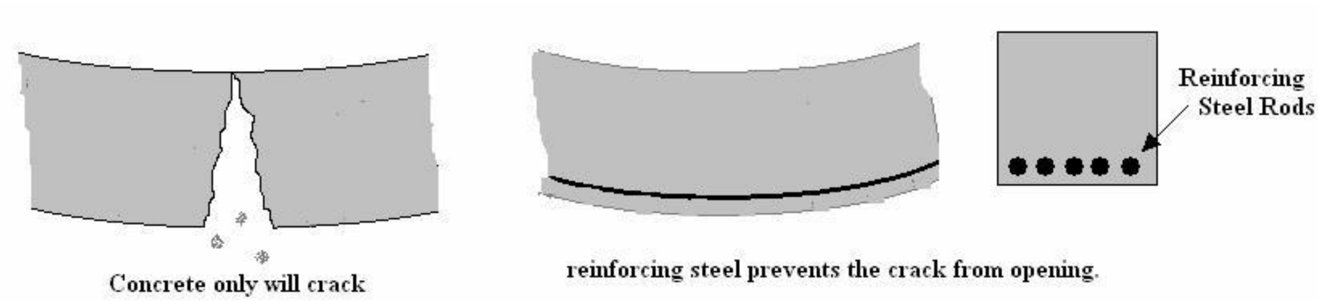
I is the second moment of area about the centroid.

$Z = I/y = 2I/D$ is the elastic modulus. (This may be looked up in standard tables for commercial beams of various sections).

$M = \sigma Z$ where σ is the actual value of stress at the outer edge so long as it is within the elastic limit.

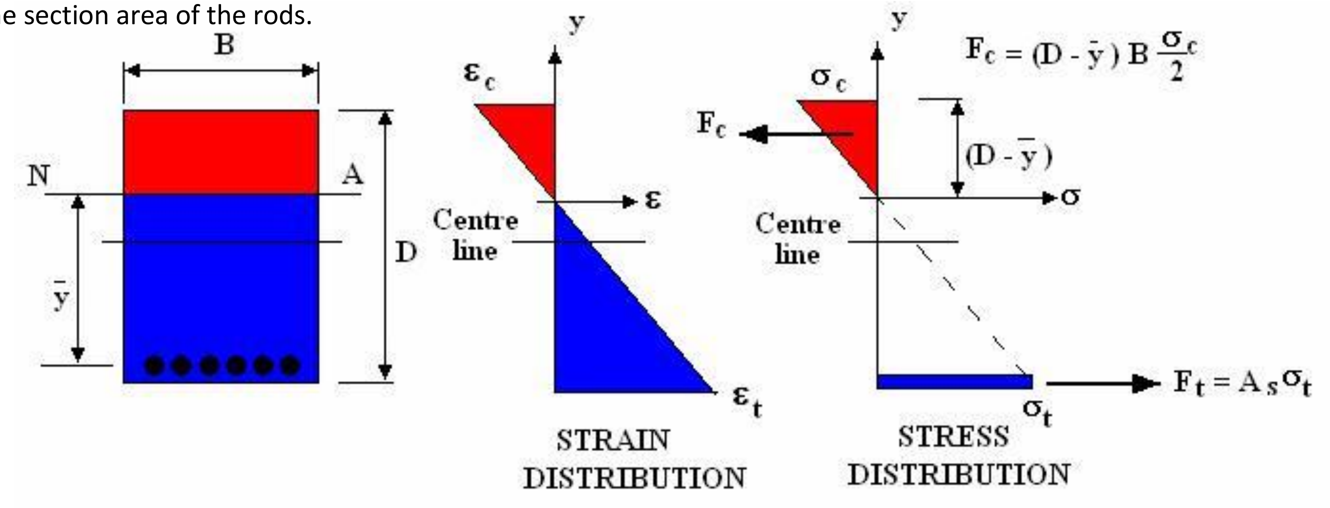
CONCRETE BEAMS

A beam made from pure concrete would fail very easily because concrete cannot withstand tension so a crack would open up on the tensile edge and spread through the section. To stop this happening, steel rods are embedded in the concrete near to the tensile edge. These are convoluted to help prevent them slipping in the concrete. We assume that the rods are so firmly embedded in the concrete that they become strained by the same amount as the concrete and although the concrete would break, the rods will bridge the gap and prevent the crack from opening

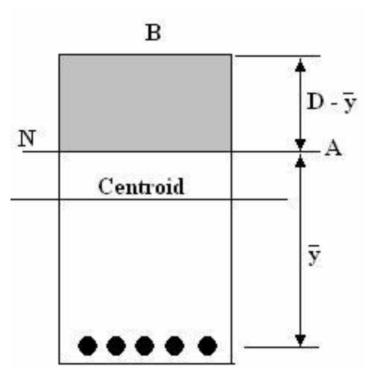


The concrete under tension does not produce any force. The entire tensile force is carried by the steel rods.

The diagram shows how the stress and strain are distributed in the concrete and steel reinforcement. As is the section area of the rods.



The problem now is that the neutral axis moves away from the centroid and to find y we need to study how to find the second moment of area when we have two materials.



SECOND MOMENT OF AREA

The concrete under compression is the shaded area and the compression force acting on this area must be equal to the tensile force in the steel rods. Because of this, the neutral axis is no longer at the middle and we need a way of determining y.

For the concrete section only, Ic = B(D - y)3/3 about the neutral Axis. Assuming the steel is confined to one level and that y is measured from this position, Is = Asy2

This is steel and not concrete and we need to find the equivalent second moment of area for the equivalent area of concrete. Consider a section of steel on which the stress is σ.

The Force is F = As σ and since σ = ε E

Note for http://sensibledewpoint.blogspot.com

The stress is $\sigma s = As\varepsilon\ Es$ for elastic materials. Suppose the steel was replaced with an area of concrete of equal strength and the same strain.

F = $Ac\varepsilon\ Ec$ = $As\varepsilon\ Es$ Ac= $As\varepsilon\ Es/\ \varepsilon\ Ec$= As Es/Ec= n As n = Es/Ec

This is the equivalent area of concrete and the equivalent second moment of area is Is = n As y2

The second moment of area for the section in equivalent concrete terms is

$$I_{NA} = \frac{B(D-\overline{y})^3}{3} + nA_s\overline{y}^2$$

Now consider the force balance again

The tensile force in the rods is Ft = $As\sigma t$ = As $Et\varepsilon t$

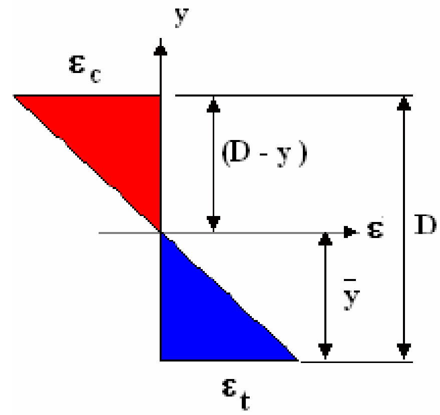
The compressive force in the concrete is (D - y) B $\sigma c/2$ = (D - y) B $Ec\varepsilon c/2$

The tensile and compressive forces must be equal. As $Et\varepsilon t$ = (D - y) B $Ec\varepsilon c/2$

$$\frac{\varepsilon_c}{\varepsilon_t} = \frac{2A_s}{B(D-\overline{y})}\frac{E_t}{E_c} = \frac{2nA_s}{B(D-\overline{y})} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

From the two similar triangles in the strain distribution we have

$$\frac{\varepsilon_c}{D-\bar{y}} = \frac{\varepsilon_t}{\bar{y}} \text{ hence } \frac{\varepsilon_c}{\varepsilon_t} = \frac{D-\bar{y}}{\bar{y}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$



Equate (1) and (2) $\quad \frac{D-\bar{y}}{\bar{y}} = \frac{2nA_s}{B(D-\bar{y})}$

This may be rearranged into a quadratic equation.

$$(D-\bar{y})^2 = \frac{2n\bar{y}A_s}{B} \qquad D^2 + \bar{y}^2 - 2D\bar{y} - \frac{2n\bar{y}A_s}{B} = 0$$

$$\bar{y}^2 - 2\bar{y}\left(D + n\frac{A_s}{B}\right) + D^2 = 0$$

Now use the quadratic formula to solve

$$\bar{y} = \frac{-b \pm \sqrt{b^2 - 4ac^2}}{2a} \qquad a = 1 \qquad b = -2\left(D + n\frac{A_s}{B}\right) \qquad c = D^2$$

$$\bar{y} = \frac{2\left(D + n\frac{A_s}{B}\right) \pm \sqrt{4\left(D + n\frac{A_s}{B}\right)^2 - 4D^2}}{2} \qquad \text{let} \left(D + n\frac{A_s}{B}\right) = \rho$$

$$\bar{y} = \frac{2\rho \pm \sqrt{4\rho^2 - 4D^2}}{2} = \rho \pm \sqrt{\rho^2 - D^2}$$

$$\bar{y} = \rho \pm \sqrt{(\rho - D)(\rho + D)}$$

WORKED EXAMPLE No.1

A rectangular reinforced concrete beam has the following data.

D = 155 mm B = 120 mm n = 8 A_s = 240 mm^2

Find the position of the neutral axis and the second moment of area about it.

SOLUTION

Suffix 'c' refers to concrete and 's' to steel.

$$\rho = \left(D + n\frac{A_s}{B} \right) = \left(155 + 8\frac{240}{120} \right) = 315$$

$$\bar{y} = 171 \pm \sqrt{(171-155)(171+155)} = 315 \pm \sqrt{(16)(326)} = 171 \pm 72.2$$

$\bar{y} = 98.8$ mm from the steel rods.

$$I_{NA} = \frac{B(D-\bar{y})^3}{3} + nA_s\bar{y}^2$$

$$I_{NA} = \frac{120(155-98.8)^3}{3} + 8 \text{ x } 240 \text{ x } 98.8^2 = 7.1 \text{ x } 10^6 + 18.742 \text{x } 10^6 = 25.842 \text{ x } 10^6 \text{ mm}^4$$

WORKED EXAMPLE No.2

For the same beam as example 1, the maximum strain in the concrete must not exceed 0.0003. Determine the maximum compressive stress in the concrete, the strain in the steel and the stress in the steel. Show that the tensile and compressive forces are equal. Assume the strain distribution is linear.

Take $E_c = 25$ GPa

SOLUTION

$E_s = n\ E_c = 8 \times 25 = 200$ GPa

$$\sigma_c = E_c \varepsilon_c = 25 \times 10^9 \times 0.0003 = 7.5 \text{ MPa}$$

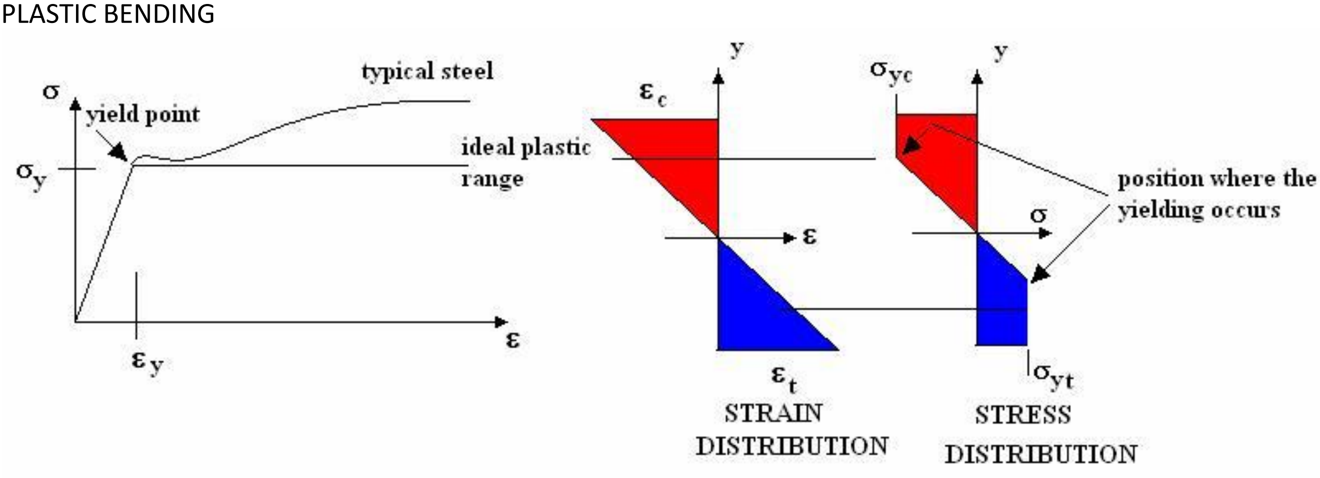
$$\varepsilon_t = \frac{\bar{y}\varepsilon_c}{D - \bar{y}} = \frac{98.8 \times 10^{-3} \times 0.0003}{155 \times 10^{-3} - 98.8 \times 10^{-3}} = 0.000527$$

$$\sigma_s = E_s \varepsilon_s = 200 \times 10^9 \times 0.000527 = 105.4 \text{ MPa}$$

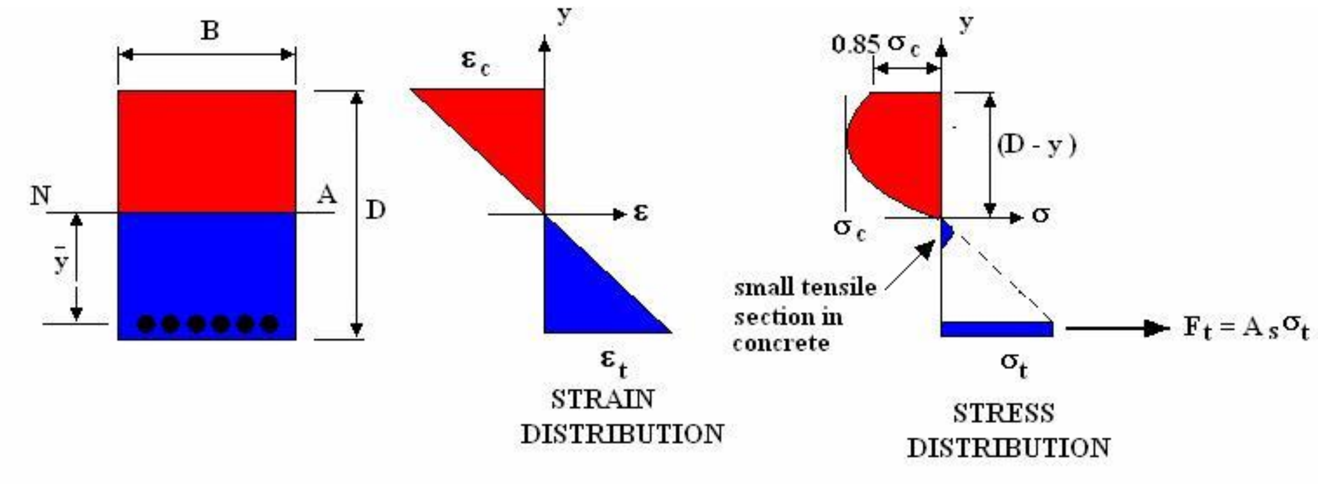
$F_c = B(D - \bar{y})\sigma_c/2 = \{12\ (155 - 98.8)\} \times 10^{-6} \times 75 \times 10^6/2 = 252.9$ kN (compressive)
$F_s = A_s\, \sigma_s/= 240 \times 10^{-6} \times 1054 \times 10^6 = 252.9$ kN (Tensile)

The work covered so far is not a very realistic approach to reinforced beams and we should consider the theory of plastic failure and true stress distribution in order to make a realistic stab at solving beam problems. Most published text on reinforced concrete beams state that concrete fails in compression when the strain is 0.003. Further studies reveal that the steel will usually have passed its tensile yield point. The type of steel used is assumed to have a plastic range and so the stress in the steel will be the yield stress. Typical steel has a yield stress of around 415 MPa and an elastic modulus E of 200 GPa. The strain at the yield point is hence around 0.002. It is important to check that the steel has yielded if the yield stress value is used.

PLASTIC BENDING

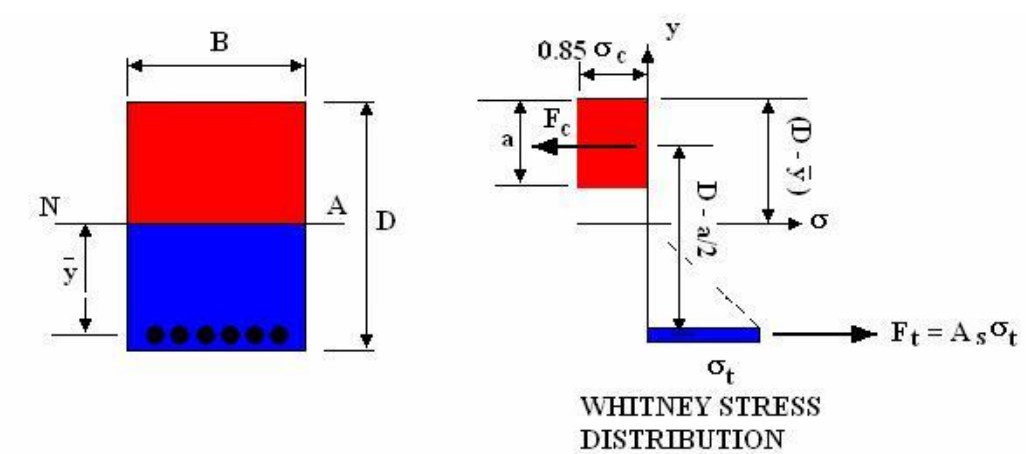


If the material reaches its yield stress at some value of y less than D/2, the stress is no longer proportional to strain. To simplify matters let's assume that the stress stays constant at the yield value σ_{yt} for all strains beyond the yield point. The diagram compares pure plastic yielding with that of typical steel. The stress and strain distribution is now as shown. Note that the strain distribution is unchanged since strain is directly proportional to the distance from the neutral axis.



It may be of interest to note that for steel beams, yielding first occurs at the outer edge and the bending moment is M = σy Zp where Zp is the plastic modulus. This may also be found in tables for commercial sections but this is not much use for reinforced beams.

The relationship between stress and strain is not linear for the concrete. Although the strain distribution remains the same as before, the stress distribution is more like this.



In order to work out the compressive force in the concrete, the stress distribution is simplified to a box

as shown. This is known as the Whitney box.

Note for http://sensibledewpoint.blogspot.com

The compressive stress is taken as a rectangle of height a and length of $0.85\sigma c$

The dimension 'a' usually given as $a = \beta 1 (D - y)$

The compressive force Fc acts at the middle of the box a distance (D-y-a/2) from the neutral axis.

It is normal to take $\beta 1 = 0.85$ for the $\sigma c \leq 30$ MPa and 0.65 for the largest stress values.

The compressive force is now given by expression $Fc = 0.85B\, a\, \sigma c$.

Equating and compressive force we have.

As $\sigma t = 0.85B\, a\, \sigma c$

$$a = \frac{A_s \sigma_t}{0.85 B \sigma_c}$$

The bending moment is $M = F_t (D - a/2) = F_c (D - a/2)$

$M = F_t (D - a/2) = A_s \sigma_s (D - a/2)$

Substitute for a and

$$M = A_s \sigma_t \left(D - \frac{A_s \sigma_t}{2 \times 0.85 B \sigma_c} \right) = A_s \sigma_t \left(D - \frac{0.59 A_s \sigma_t}{B \sigma_c} \right)$$

- The tensile strength in the concrete is always ignored.
- The steel rods may have yielded in which case the $\sigma t = \sigma yt$
- Concrete is assumed to have failed in compression when the strain reaches 0.003.
- The strain distribution is assumed to be linear.
- There are limitations to the formulae developed here and more advanced text should be studied before applying them to real situations.